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# A branch point in the critical surface of the Ashkin–Teller model in the renormalization group theory

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**Abstract.** It is shown that a branch point appears in the critical surface of the Ashkin–Teller model at the point where this model reduces to a Potts model, and its relation with the eight-vertex model is discussed. An estimate for the critical exponent $\alpha$ of the Potts model is given.

## 1. Introduction

Recently the symmetric eight-vertex model has received much attention (Nauenberg and Nienhuis 1974, van Leeuwen 1975) in the renormalization group theory since its continuously varying exponents, arising (as conjectured by Kadanoff and Weger 1971) from the existence of a line of fixed points, offer a special challenge to the theory. In this note a renormalization group study is made of the model of Ashkin and Teller (1943) which is in some sense closely related to the eight-vertex model and can in fact be identified with a more general so called 'staggered' eight-vertex model (Fan 1972, Wegner 1972). However, it will be seen that the critical behaviour of the Ashkin–Teller model (ATM) differs from that of the normal eight-vertex model by the appearance of a branch point in the critical surface as a consequence of the existence of two transition temperatures in the ATM. This fact which was originally overlooked (Fan 1972) was also recently noted by Wu and Lin (1974).

## 2. The Ashkin–Teller model

The model under study here is in fact a special case of the ATM. Consider a square lattice having at every site a (two-dimensional) spin $\boldsymbol{S}$ which can point along the four directions of the coordinate axes. Define the Hamiltonian

$$H = -\sum_{\langle i,j\rangle} \{J(\boldsymbol{S}_i\,.\,\boldsymbol{S}_j)+K[2(\boldsymbol{S}_i\,.\,\boldsymbol{S}_j)^2-1]\} \tag{1}$$

where $\langle i,j\rangle$ denotes summation over nearest neighbours. This model is an isotropic version of the ATM. The following facts are helpful in understanding some of the critical properties of the model.

(i) If $J = 0$ the model reduces essentially to a two-component Ising model with spins that order either along the $x$ axis or along the $y$ axis; the model is critical for $e^{-2K} = \sqrt{2}-1$.

(ii) If $K = +\infty$, only configurations with spins which are parallel or antiparallel remain, so that again an Ising model appears which is critical for $e^{-2J} = \sqrt{2}-1$.

(iii) One can represent the orientations of $S$ by two Ising spins $(\sigma, \tau)$ in the following way: $\sigma = S_x - S_y$, $\tau = S_x + S_y$. Note that

$$(\boldsymbol{S}_i \,.\, \boldsymbol{S}_j) = \tfrac{1}{2}\sigma_i\sigma_j + \tfrac{1}{2}\tau_i\tau_j,$$

hence

$$H = -\sum_{\langle i,j\rangle} (\tfrac{1}{2}J\sigma_i\sigma_j + \tfrac{1}{2}J\tau_i\tau_j + K\sigma_i\sigma_j\tau_i\tau_j). \tag{2}$$

When $K = 0$, the model reduces to two independent Ising models with nearest-neighbour coupling $\frac{1}{2}J$ and a critical point at $e^{-J} = \sqrt{2}-1$. The coupling $K$ couples the two Ising lattices (which can be pictured as a two-layered square lattice) with a four-body interaction. The spin representation (Kadanoff and Wegner 1971, Wu 1971) of the eight-vertex model is also of the type of two Ising models coupled by a four-body interaction; however, the topology of this interaction is different (Fan 1972).

(iv) The energy levels of mutually orthogonal spins and those of antiparallel spins cross each other at the line $K = \frac{1}{2}J$ and the model reduces to a four-component Potts model (Potts 1952) with a critical point given by $e^{-2J} = \frac{1}{3}$.

(v) There exists a symmetry under a duality transformation for the ATM (Mittag and Stephen 1971) which can be formulated in terms of the relative weights $b = \exp(-2J)$ and $c = \exp(-2K-J)$ as

$$b' = \frac{1+b-2c}{1+b+2c}, \qquad c' = \frac{1-b}{1+b+2c}. \tag{3}$$

The line $b+2c = 1$ is invariant under this transformation, and if there existed only one critical temperature this line would be the critical line. The Potts critical point and the critical point at $K = 0$ are both on this line. However, this line does not pass through the critical points found in (i) and (ii); these points are mapped into each other by the duality transformation. Symmetry considerations lead then to the conjecture that the critical line coincides with the line invariant under duality up to the Potts critical point, after which it branches into two curves which are mapped into each other by duality as shown in figure 1.

It is remarkable that the eight-vertex model with weights $a, b, c, d$ (in the notation of Kadanoff and Wegner 1971) is invariant under the same duality transformation as the ATM (Fan 1972). In particular, in case $c = d$ this duality transformation in terms of the relative weights $b, c$ (obtained by setting $a = 1$) is given again by (3). In the case of the eight-vertex model, however, the critical line coincides with the line $b + 2c = 1$ for $b > 0$, as can be seen from Baxter's famous exact solution (Baxter 1971).

Further evidence in favour of a critical surface, shaped according to figure 1, is given by Wu and Lin (1974). They note that the mapping which, in general, transforms the ATM into a staggered eight-vertex model happens to map the line $b_{\mathrm{AT}}+2c_{\mathrm{AT}} = 1$ into the line $b_{8\mathrm{v}}+2c_{8\mathrm{v}} = 1$ of the normal eight-vertex model (with $c = d$) in such a way that

$$b_{8\mathrm{v}} = \frac{1-3b_{\mathrm{AT}}}{3-b_{\mathrm{AT}}}. \tag{4}$$

Consequently the segment $\frac{1}{3} < b_{\mathrm{AT}} < 1$ of the line $b_{\mathrm{AT}}+2c_{\mathrm{AT}} = 1$ corresponds to negative values of $b_{8\mathrm{v}}$ on the line $b_{8\mathrm{v}}+2c_{8\mathrm{v}} = 1$ which are *not* critical according to Baxter's solution.

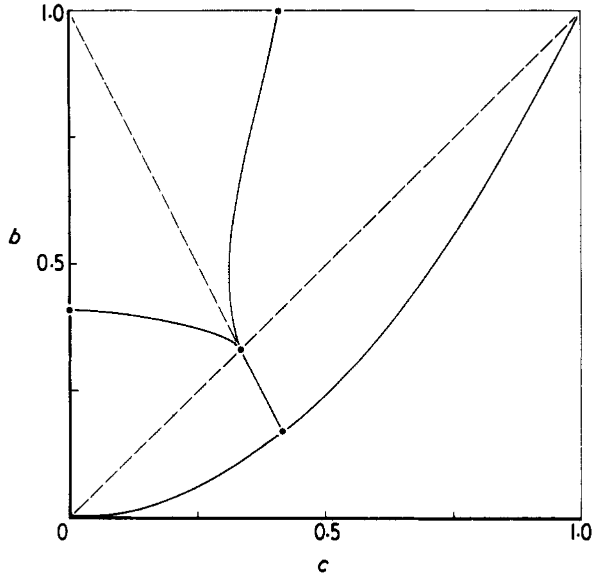


**Figure 1.** Schematic plot of the critical lines for the ATM as expected from general considerations. The line $b = c$ is the Potts axis.

## 3. Renormalization group transformation

A renormalization group transformation on the ATM can be defined by a slight generalization of the method of Niemeyer and van Leeuwen (1974) developed for Ising models. Consider a unit square as an elementary cell and define a cell spin $\boldsymbol{S}$ associated with the configurations of the four site spins $\boldsymbol{S}_i$ according to the following rules:

(i) Associate with $\boldsymbol{S}'$ all configurations for which the majority of spins point along $\boldsymbol{S}'$.

(ii) Associate with $\boldsymbol{S}'$ with a weight $W = \frac{1}{2}$ all configurations for which two spins point along $\boldsymbol{S}'$ and the remaining spins point in one other direction.

(iii) Associate with $\boldsymbol{S}'$ with a weight $W = \frac{1}{4}$ all configurations for which all spins have different directions.

Notice that this choice of $\boldsymbol{S}'$ is not the only one possible, the present choice being motivated by symmetry considerations†. The renormalization group transformation is now defined by

$$\exp(-H'\{\boldsymbol{S}'_j\}) = \sum_{\boldsymbol{S}_{j,i}} \prod_j W(\boldsymbol{S}'_j, \boldsymbol{S}_{j,i}) \exp(-H\{\boldsymbol{S}_{j,i}\}) \tag{5}$$

where $j$ numbers the cells and $i = 1, 4$ numbers the sites in cell $j$.

In first order in the cumulant expansion (Niemeyer and van Leeuwen 1974) new couplings are not generated and the renormalization group transformation takes the form

$$\begin{aligned} J' &= 2J\left(\frac{e^{4J+4K} + 9\,e^{2J} + 2\,e^{4K} + 12}{e^{4J+4K} + 14\,e^{2J} + 7\,e^{4K} + 36 + 6\,e^{-2J}}\right)^2 \\ K' &= 2K\left(\frac{e^{4J+4K} + 4\,e^{2J} + 7\,e^{4K} + 12}{e^{4J+4K} + 14\,e^{2J} + 7\,e^{4K} + 36 + 6\,e^{-2J}}\right)^2. \end{aligned} \tag{6}$$

† Another possible choice of $\boldsymbol{S}'$ based on the total cell magnetization would only imply invariance under rotations, whereas the present choice is invariant under arbitrary permutations.

## 4. Discussion

Investigation of these transformations for $J$ and $K$ positive, in the parameter space of the relative weights $b = \exp(-2J)$ and $c = \exp(-2K-J)$, leads to the following conclusions.

The transformation possesses a total of seven fixed points of which three are stable, three are stable in one direction, and one, which is the fixed point located on the Potts axis, is unstable. A repeated application of the transformation (6) brings (almost all) points of the parameter space close to one of the stable fixed points; this leads to a division of the parameter space into three regions, each attracted by one of the stable fixed points. The ridge lines which are the boundaries between these regions are the critical lines (Wilson 1972); each of them runs from the unstable fixed point on the Potts axis to one of the one-sided stable fixed points, giving rise to a branching of the critical surface at the Potts axis (see figure 2). Notice that the critical lines which meet at the branch point are all tangential to a direction which corresponds to the eigenvector with smallest eigenvalue at the unstable fixed point.

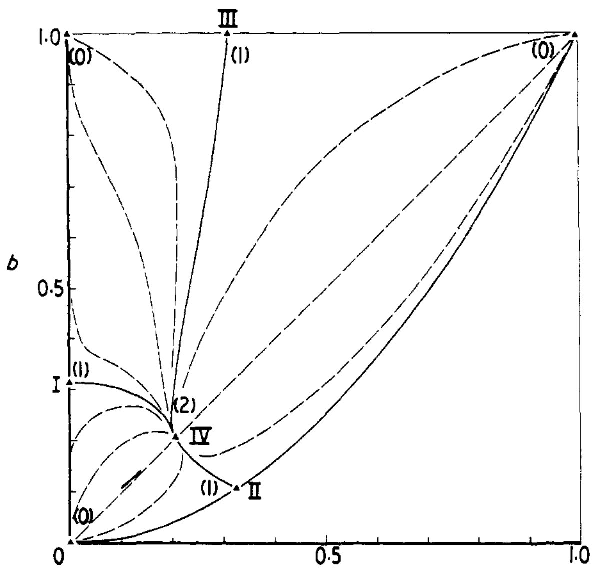


**Figure 2.** Critical lines for the ATM obtained from the first-order approximation to the renormalization group equations. ▲ represent fixed points; the number of eigenvalues larger than one is given in brackets. Broken lines correspond to arbitrary flow paths.

The critical exponent $\alpha$ can be obtained from the eigenvalue $\lambda$ by the relation $\alpha = 2-d \ln l/\ln \lambda$ (where $d$ is the dimension and $l$ the scaling length). The results for the square and triangular lattices are collected in table 1. The fixed points which are one-sided stable are expected to be of Ising type and should therefore have $\alpha = 0$; the values as obtained in first order deviate from this value by amounts normally found in this order. The fixed point on the Potts axis should give information on the critical behaviour of the Potts model of which little is known exactly. There is an interesting paper by Baxter (1973) on the $n$-component Potts model in which he finds that the transition is of first order for $n > 4$, which is, incidentally, the mean-field result for $n \geqslant 3$, and of higher

**Table 1.** Values of the critical exponent $\alpha$ in first-order cumulant expansion. Roman numerals identify the fixed points in this table with the corresponding ones in figure 2.

| Fixed point | Square | Triangular |
|---|---|---|
| $K = \infty$ (I) | 0·17 | −0·25 |
| $K = 0$ (II) | 0·06 | 0·18 |
| $J = 0$ (III) | 0·17 | 0·15 |
| $K = \frac{1}{2}J$ (IV) | 0·46 | 0·45 |

order for $n < 4$. The present estimate of $\alpha = 0{\cdot}5$ confirms this prediction for $n > 4$.

An interesting question remains whether $\alpha$ will change continuously along the critical line which connects the Potts critical point with the critical point at $K = 0$. It is this line which, as already noted, can be mapped into the eight-vertex model where $\alpha$ does change continuously; however, no firm conclusions can be drawn from this equivalence since the temperature directions in the two models do not correspond. What can be said is that the same argument as presented by Kadanoff and Wegner (1971) (compare also van Leeuwen 1975) for the existence of a marginal operator at $K = 0$ holds also in the present case. The presence of such an operator at a fixed point is only a necessary but no sufficient condition for the existence of a *line* of fixed points, so that the question of the existence of a fixed line and the continuously varying exponents associated with it can as yet not be solved for the ATM. The perturbation theory does not easily yield clear indications for the existence of a fixed line; at the fixed point with $K = 0$ the eigenvalue along the direction which theoretically should be marginal (ie $\lambda = 1$) is still rather far from unity.

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